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Original Article: TOWARDS FLUID DYNAMICS EQUATIONS

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Abstract
Some new formulas and results on transformation of hydrodynamics equations are presented. New problems for improvement of fluid dynamics calculation methods are given.

The search for analytic solutions of the equations of hydrodynamics and the study of their properties are of great interest [1]. One of the classic techniques is to convert variables and unknown functions in order to simplify the system of equations of hydrodynamics and convert to a form suitable for modeling or further study. Recently, there have been received new integrals of the Euler equations [2]. Here we present them to convert the Euler equations in infinite space to the form convenient for calculations.

1. Euler equations. Euler equations of incompressible fluid in infinite space $(\mathbf{r},t)$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times [\mathbf{u} \times \boldsymbol{\omega}], \quad (1)$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}, \quad (2)$$

$$\mathrm{div}\mathbf{u} = 0 \quad (3)$$

give the following integrals of motion [1, 2]

$$\boldsymbol{\omega} \cdot \nabla x_0 = \omega_{0x}(x_0, y_0, z_0), \quad (4)$$

$$\boldsymbol{\omega} \cdot \nabla y_0 = \omega_{0y}(x_0, y_0, z_0), \quad (5)$$

$$\boldsymbol{\omega} \cdot \nabla z_0 = \omega_{0z}(x_0, y_0, z_0), \quad (6)$$

$$\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}=1, \qquad (7)$$

$$\frac{d\mathbf{r}}{dt}=\mathbf{u}\,,\ \mathbf{r}=\mathbf{r}(\mathbf{r}_0,t)\,,\ \mathbf{r}_0=\mathbf{r}_0(\mathbf{r},t)\,, \qquad (8)$$

where $(\mathbf{r}_0,t)$ Lagrange variables. One follows the following expressions from them

$$\omega_x(x_0,y_0,z_0,t)=\boldsymbol{\omega}_0\cdot\nabla_0 x\,, \qquad (9)$$

$$\omega_y(x_0,y_0,z_0,t)=\boldsymbol{\omega}_0\nabla_0 y\,, \qquad (10)$$

$$\omega_z(x_0,y_0,z_0,t)=\boldsymbol{\omega}_0\cdot\nabla_0 z\,, \qquad (11)$$

where $\nabla_0=(\partial/\partial x_0,\partial/\partial y_0,\partial/\partial z_0)$. One has the following integral expression to velocity $\mathbf{u}$

$$\mathbf{u}=\nabla_{\mathbf{r}}\times\int\frac{\boldsymbol{\omega}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'=-\int\frac{(\mathbf{r}-\mathbf{r}')\times\boldsymbol{\omega}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}'\,. \qquad (12)$$

We proceed in the integrand of (12) to the variables of Lagrange. Then, using (7), (9) - (11), we find

$$\mathbf{u}=-\int\frac{(\mathbf{r}-\mathbf{r}')\times\boldsymbol{\omega}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}'=-\int\frac{(\mathbf{r}-\mathbf{r}'(\mathbf{r}_0',t))\times\boldsymbol{\omega}(\mathbf{r}_0',t)}{|\mathbf{r}-\mathbf{r}'(\mathbf{r}_0',t)|^3}\frac{\partial\mathbf{r}'}{\partial\mathbf{r}_0'}d^3\mathbf{r}_0'=$$
$$=-\int(\mathbf{r}-\mathbf{r}'(\mathbf{r}_0',t))\times\frac{(\boldsymbol{\omega}_0(\mathbf{r}_0')\cdot\nabla_0)\mathbf{r}'(\mathbf{r}_0',t)}{|\mathbf{r}-\mathbf{r}'(\mathbf{r}_0',t)|^3}d^3\mathbf{r}_0'\,. \qquad (13)$$

Now, given the definition of the Lagrangian variables (8), we have

$$\mathbf{u}(\mathbf{r}_0,t)=\frac{\partial\mathbf{r}}{\partial t_L}(\mathbf{r}_0,t)=-\int(\mathbf{r}(\mathbf{r}_0,t)-\mathbf{r}'(\mathbf{r}_0',t))\times\frac{(\boldsymbol{\omega}_0(\mathbf{r}_0')\cdot\nabla_0)\mathbf{r}'(\mathbf{r}_0',t)}{|\mathbf{r}(\mathbf{r}_0,t)-\mathbf{r}'(\mathbf{r}_0',t)|^3}d^3\mathbf{r}_0' \qquad (14)$$

Thus, we have a system of integral-differential equations (14) to determine the $\mathbf{r}(\mathbf{r}_0,t)$, and hence by (8) and the entire flow. This system of equations is conveniently modeled as higher derivatives with respect to time expressed in terms of the others. For this system, the Cauchy initial data are defined as follows

$$\mathbf{r}|_{t=0}=\mathbf{r}_0\,. \qquad (15)$$

But the initial velocity distribution $\mathbf{u}_0(\mathbf{r})$ is already presented in the structure of the equations (14).

Laplace equation is important in the theory of combustion, flow problems, potential theory, etc. Properties of solutions of the Laplace equation are well known. Here we present another property, apparently resulting from the known facts.

2. Laplace equation. Consider a two-dimensional Laplace equation in a volume G [1]

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 . \qquad (16)$$

The values $\partial\varphi/\partial n$ and φ on the boundary of G are linked by Green's formula [3]

$$\pi\varphi(\mathbf{r})=\oint\left[\frac{\partial\varphi(\mathbf{r}_s)}{\partial n_s}\ln|\mathbf{r}_s-\mathbf{r}|-\varphi(\mathbf{r}_s)\mathbf{n}_s\cdot\frac{\mathbf{r}_s-\mathbf{r}}{|\mathbf{r}_s-\mathbf{r}|^2}\right]dl(\mathbf{r}_s), \qquad (17)$$

where the index s is integrated across the border. In G holds, obviously, also the equation

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial\varphi}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial\varphi}{\partial x}\right)=0 . \qquad (18)$$

Suppose that function φ is smooth enough in G. It is also possible to write down the Green's formula [3]

$$\pi\frac{\partial\varphi}{\partial x}(\mathbf{r})=\oint\left[\frac{\partial\left(\frac{\partial\varphi}{\partial x}\right)(\mathbf{r}_s)}{\partial n_s}\ln|\mathbf{r}_s-\mathbf{r}|-\frac{\partial\varphi}{\partial x}(\mathbf{r}_s)\mathbf{n}_s\cdot\frac{\mathbf{r}_s-\mathbf{r}}{|\mathbf{r}_s-\mathbf{r}|^2}\right]dl(\mathbf{r}_s) . \qquad (19)$$

Transform this formula, using the obvious relations at the border (see. Fig.)

$$\frac{\partial\varphi}{\partial x}=\left(n_x\frac{\partial\varphi}{\partial n}+\tau_x\frac{\partial\varphi}{\partial\tau}\right) \qquad (20)$$

and

$$\frac{\partial^2\varphi}{\partial n^2}+\frac{\partial^2\varphi}{\partial\tau^2}=0 . \qquad (21)$$

We have, after substituting (20) into (19),

$$\pi\left(n_x\frac{\partial\varphi}{\partial n}+\tau_x\frac{\partial\varphi}{\partial\tau}\right)(\mathbf{r})=$$

$$=\oint\left[\frac{\partial\left(n_{xs}\frac{\partial\varphi}{\partial n_s}+\tau_{xs}\frac{\partial\varphi}{\partial\tau_s}\right)(\mathbf{r}_s)}{\partial n_s}\ln|\mathbf{r}_s-\mathbf{r}|-\left(n_{xs}\frac{\partial\varphi}{\partial n_s}+\tau_{xs}\frac{\partial\varphi}{\partial\tau_s}\right)(\mathbf{r}_s)\mathbf{n}_s\cdot\frac{\mathbf{r}_s-\mathbf{r}}{|\mathbf{r}_s-\mathbf{r}|^2}\right]dl(\mathbf{r}_s) \qquad (22)$$

or

$$\pi\left(n_x\frac{\partial\varphi}{\partial n}+\tau_x\frac{\partial\varphi}{\partial\tau}\right)(\mathbf{r})=$$

$$= \oint \left[\left(n_{xs} \frac{\partial^2 \varphi}{\partial n_s^2} + \tau_{xs} \frac{\partial^2 \varphi}{\partial n_s \partial \tau_s} \right)(\mathbf{r}_s) \ln|\mathbf{r}_s - \mathbf{r}| - \left(n_{xs} \frac{\partial \varphi}{\partial n_s} + \tau_{xs} \frac{\partial \varphi}{\partial \tau_s} \right)(\mathbf{r}_s) \mathbf{n}_s \cdot \frac{\mathbf{r}_s - \mathbf{r}}{|\mathbf{r}_s - \mathbf{r}|^2} \right] dl(\mathbf{r}_s) \qquad (23)$$

or, taking into account (21),

$$\pi \left(n_x \frac{\partial \varphi}{\partial n} + \tau_x \frac{\partial \varphi}{\partial \tau} \right)(\mathbf{r}) =$$

$$= \oint \left[\left(-n_{xs} \frac{\partial^2 \varphi}{\partial \tau_s^2} + \tau_{xs} \frac{\partial^2 \varphi}{\partial n_s \partial \tau_s} \right)(\mathbf{r}_s) \ln|\mathbf{r}_s - \mathbf{r}| - \left(n_{xs} \frac{\partial \varphi}{\partial n_s} + \tau_{xs} \frac{\partial \varphi}{\partial \tau_s} \right)(\mathbf{r}_s) \mathbf{n}_s \cdot \frac{\mathbf{r}_s - \mathbf{r}}{|\mathbf{r}_s - \mathbf{r}|^2} \right] dl(\mathbf{r}_s). \qquad (24)$$

So we got two integral-differential equations (17) and (24) on the boundary of G for the two unknown on the surface $\partial\varphi/\partial n$ and φ. Obviously, they still need to set the boundary conditions at a certain point on the boundary G. If we consider the three-dimensional Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \qquad (25)$$

then the quantities $\partial\varphi/\partial n$ and φ are connected by Green's formula

$$2\pi\varphi(\mathbf{r}) = \oiint \left[\frac{1}{|\mathbf{r}_s - \mathbf{r}|} \frac{\partial \varphi(\mathbf{r}_s)}{\partial n_s} + \varphi(\mathbf{r}_s) \mathbf{n}_s \cdot \frac{\mathbf{r}_s - \mathbf{r}}{|\mathbf{r}_s - \mathbf{r}|^3} \right] dS(\mathbf{r}_s), \qquad (26)$$

where the index s denotes integration over the entire surface. Moreover, we can get similar results.

Of great interest to researchers is the Navier-Stokes equations. Meanwhile, little attention has paid to the behavior of their solutions when you change the dimensions of these equations. It turns out that it is possible to convert them to linear ones by increasing the number of variables. Furthermore, in [5, 6] it is shown as possible, then again reducing the number of variables in the systems of differential equations, keeping their linearity. Here we present these simple ideas.

3. The Navier-Stokes equations. Consider the Navier-Stokes equations in two dimensions in the form [1]

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + \frac{\partial P}{\partial x} = \nu \Delta u_x, \qquad (27)$$

$$\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + \frac{\partial P}{\partial y} = \nu \Delta u_y, \qquad (28)$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0. \qquad (29)$$

Obviously, they can be transformed into [1]

$$\frac{\partial u_x}{\partial t} + \frac{\partial (u_x)^2}{\partial x} + \frac{\partial (u_x u_y)}{\partial y} + \frac{\partial P}{\partial x} = \nu \Delta u_x, \qquad (30)$$

$$\frac{\partial u_y}{\partial t}+\frac{\partial\left(u_y u_x\right)}{\partial x}+\frac{\partial\left(u_y\right)^2}{\partial y}+\frac{\partial P}{\partial y}=v\Delta u_y\ , \tag{31}$$

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0\ . \tag{32}$$

We introduce the additional function

$$U(\mathbf{r},\mathrm{t},\xi)=P(\mathbf{r},\mathrm{t})+\left(u_y u_x\right)\xi+\left(u_x\right)^2\frac{\xi^2}{2!}+\left(u_y\right)^2\frac{\xi^3}{3!}\ . \tag{33}$$

Then

$$\frac{\partial U}{\partial\xi}(\mathbf{r},\mathrm{t},\xi)=\left(u_y u_x\right)+\left(u_x\right)^2\xi+\left(u_y\right)^2\frac{\xi^2}{2!}\ , \tag{34}$$

$$\frac{\partial^2 U}{\partial\xi^2}(\mathbf{r},\mathrm{t},\xi)=\left(u_x\right)^2+\left(u_y\right)^2\xi\ , \tag{35}$$

$$\frac{\partial^3 U}{\partial\xi^3}(\mathbf{r},\mathrm{t},\xi)=\left(u_y\right)^2\ . \tag{36}$$

Consequently,

$$\left(u_y\right)^2=\frac{\partial^3 U}{\partial\xi^3}\ , \tag{37}$$

$$\left(u_x\right)^2=\frac{\partial^2 U}{\partial\xi^2}-\frac{\partial^3 U}{\partial\xi^3}\xi\ , \tag{38}$$

$$\left(u_y u_x\right)=\frac{\partial U}{\partial\xi}-\frac{\partial^2 U}{\partial\xi^2}\xi+\frac{\partial^3 U}{\partial\xi^3}\frac{\xi^2}{2}\ , \tag{39}$$

$$P=U-\frac{\partial U}{\partial\xi}\xi+\frac{\partial^2 U}{\partial\xi^2}\frac{\xi^2}{2}-\frac{\partial^3 U}{\partial\xi^3}\frac{\xi^3}{6}\ . \tag{40}$$

After substituting (37) - (40) in (30) - (32) we have,

$$\frac{\partial u_x}{\partial t}+\frac{\partial U}{\partial x}-\xi\frac{\partial^2 U}{\partial x\partial\xi}+\frac{\partial^2 U}{\partial y\partial\xi}+\left(1+\frac{\xi^2}{2}\right)\frac{\partial^3 U}{\partial x\partial\xi^2}-\xi\frac{\partial^3 U}{\partial y\partial\xi^2}-\left(\xi+\frac{\xi^3}{6}\right)\frac{\partial^4 U}{\partial x\partial\xi^3}+\frac{\xi^2}{2}\frac{\partial^4 U}{\partial y\partial\xi^3}=v\Delta u_x\ , \tag{41}$$

$$\frac{\partial u_y}{\partial t}+\frac{\partial U}{\partial y}-\xi\frac{\partial^2 U}{\partial y\partial\xi}+\frac{\partial^2 U}{\partial x\partial\xi}-\xi\frac{\partial^3 U}{\partial x\partial\xi^2}+\frac{\xi^2}{2}\frac{\partial^3 U}{\partial y\partial\xi^2}+\frac{\xi^2}{2}\frac{\partial^4 U}{\partial x\partial\xi^3}+\left(1-\frac{\xi^3}{6}\right)\frac{\partial^4 U}{\partial y\partial\xi^3}=v\Delta u_y\ , \tag{42}$$

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0\,. \tag{43}$$

Thus, any solution of equations (27) - (29) is contained in the solution of (41) - (43). The system (41) - (43) is linear, but in contrast to (27) - (29) contains one more than the number of variables. The correctness of the systems (41) - (43) can be studied by Cauchy-Kovalevskaya theorem [4]. To do this, you must make an arbitrary change of coordinates and express clearly higher derivatives of all quantities in terms of the others.

As it can be seen, in the system (41) - (43) the initial data must be specified on the surface, the dimension of which is larger than for the system (27) - (29). It is known that the general solution of the linear system of partial differential equations is given in form of a linear operator on the initial data [4]. Let

$$\left(u_x,u_y,U\right)(\mathbf{r},\mathrm{t},\xi)=\hat{L}\left(u^0{}_x,u^0{}_y,U^0\right). \tag{44}$$

The solution of system (27) - (29) must be contained in (44). Thus, if we define a certain surface and set $\left(u_x,u_y,U\right)$ on it in the space $(\mathbf{r},\mathrm{t},\xi)$, according to the formula (44) we find a solution in $(\mathbf{r},\mathrm{t},\xi)$. It can be shown that the surface $\{\mathrm{t}=0\}$ is characteristic of Eqs (41) - (43). It means that the initial data can not be set on this surface. However, one can specify the initial data near to its surface and get close results.

Equations (41) - (43) can be overdetermined by additional linear equation for an example

$$\frac{\partial^4 U}{\partial \xi^4}=0\,. \tag{45}$$

It is known that in this case one can reduce the dimensionality in Eqs (41) - (43) and (45) [5,6]. The reduced dimension equations will be also linear. Actually, it is proposed a method for converting the Navier-Stokes equations in a volume to the closed system of linear equations on any surface by increasing number of unknowns. Writing out of the additional function in the form (33) as well as an additional relation in the form (45) is not important.

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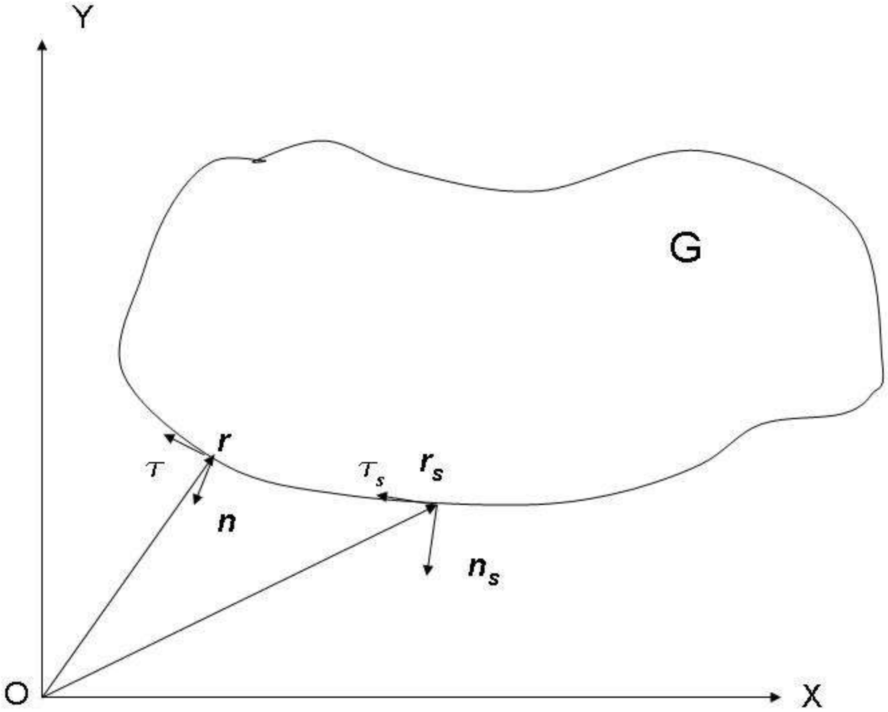

Fig. Volume G